



THE STORY OF MATH

A VIEWER'S GUIDE

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EPISODE 1 THE LANGUAGE OF THE UNIVERSE

Highlights

- The ancient Babylonians and Egyptians developed surprisingly sophisticated mathematics to solve practical problems, such as measuring weights, lengths, areas, and volumes for commerce, construction, and taxation.
- The Greeks transformed math from a system for counting and measuring to a true analytical discipline, searching for properties and patterns within numbers.
- The Greek mathematician Pythagoras codified not only the relationships among the sides of right triangles (in the famous theorem that still bears his name), but also the whole-number ratios in the intervals between musical notes.
- Euclid's *Elements*, one of the world's most important mathematical texts, catalogued formulas for finding the volumes of geometric solids and systems of mathematical proof.
- In addition to contriving many practical applications for math, Archimedes devised formulas for calculating the area of shapes, including circles—approaching the idea of π .

Questions to Consider

1. How does math help us solve problems in our daily lives?
2. Why did Plato refer to mathematics as the single most important form of philosophy? In what sense are mathematical inquiries philosophical?

EPISODE 2 THE GENIUS OF THE EAST

Highlights

- For the ancient Chinese, numbers had both mystical and practical

MATH MAGIC IN ART

16	3	2	13
5	x	11	8
9	6	y	12
z	15	14	1

The 16th-century German artist Albrecht Dürer used mathematical principles in his work—but nowhere more obviously than in *Melancholia I*. The engraving includes this grid, shown here with three numbers from the original missing. Can you find x , y , and z ?
(See page 15 for the answer.)

purposes. In addition to a complex system of numerology, they used decimals for quick calculations, discovered patterns employed in modern Internet cryptography, and estimated solutions to cubic equations centuries before Newton.

- Indian thinkers gave us the concepts of true zero, negative numbers, and infinity. They also further developed trigonometry—a way to translate geometry into numbers and back again, essential for calculating large distances.
- Arab mathematicians left a lasting legacy: in addition to algebra, they provided the very numerals that we use today, which have their roots in Hindu-Arabic numbers.
- In the 13th century, Leonardo of Pisa (alias Fibonacci) helped Europeans adopt Eastern mathematical ideas, such as the Arabic number system and the concept of zero. He also discovered the famed Fibonacci sequence, uncovering mathematical patterns in nature.

Questions to Consider

1. Why do you think the concept of zero as more than a placeholder was so important to the development of mathematics?
2. Obviously, daisies and pineapples don't know math. Then why do their growth patterns follow the Fibonacci sequence?

EPISODE 3

THE FRONTIERS OF SPACE

Highlights

- In 17th century France, René Descartes linked algebra and geometry by describing curved lines as equations.
- Working independently, Englishman Isaac Newton and German Gottfried Leibniz developed calculus—essential for mathematically describing fluids, orbiting planets, and other bodies in motion.
- In the 18th century, Leonhard Euler originated topology, did groundbreaking work in prime numbers, created many current

mathematical notations, and advanced a theorem about calculating infinite sums. Only a little later, Carl Friedrich Gauss invented modular arithmetic (a new way to perform equations that proved essential to modern number theory).

- By the mid-19th century, mathematicians such as János Bolyai in Transylvania and Bernhard Riemann in Germany began stretching geometry to describe space beyond three dimensions.

Questions to Consider

1. How would you decide the dispute between Newton and Leibniz over who invented calculus first?
2. As Du Sautoy points out, Napoleonic France emphasized utilitarian advances in mathematics, while the German states valued “pure” math—mathematics for its own sake. How does conceptual mathematics serve society?

EPISODE 4

TO INFINITY AND BEYOND

Highlights

- In the late 19th and early 20th centuries, German mathematician Georg Cantor became the first to explore the concept of infinity with mathematical rigor and precision, proving different types and sizes of infinities.
- Trying to determine whether the planets in the solar system have stable orbits, Frenchman Henri Poincaré laid the foundation for chaos theory. A way of looking at systems with a multitude of variables, chaos theory has applications in fields from climate study to medical technology. Poincaré also built on Euler's foundations to create algebraic topology, or “bendy geometry,” which describes the shapes of morphing three-dimensional objects.
- Austrian Kurt Gödel revealed uncertainty at the heart of math with his incompleteness theorem, which showed that any logical system

for math produces some true but unprovable statements.

- Attempting to prove Cantor's continuum hypothesis, American Paul Cohen demonstrated the possibility of conflicting but equally consistent answers to the same problem.
- Mathematicians such as Évariste Galois and Alexander Grothendieck pioneered new forms of math based not on numbers or shapes, but on the hidden structures of equations.

Questions to Consider

1. Cantor proved that the infinite set of whole numbers is smaller than the infinite set of decimals. Does this change your own concept of infinity?
2. In his early work, David Hilbert showed that a finite set of equations spawn the infinite number of equations in mathematics—even though he couldn't construct the originating set of equations. Critics claimed that, in proving that something must exist without actually producing it, he strayed from mathematics into theology. Why did they make that claim? Do you agree or disagree?

- After a chance meeting with physicist Freeman Dyson, mathematician Hugh Montgomery learned that the occurrence of Riemann's zeros matches the pattern of energy levels of excited nuclei.
- Although the Riemann hypothesis remains unproven, most mathematicians now accept it as true.

Questions to Consider

Overall, what do you think of the search to prove the Riemann hypothesis—is it a heroic intellectual quest, a rarefied but essentially irrelevant exercise, or something else entirely? Why do you think the Clay Mathematics Institute still offers \$1 million for its solution?

BONUS

THE MUSIC OF THE PRIMES

Highlights

- Since Euclid's time, mathematicians have sought a pattern by which to predict the occurrence of prime numbers (natural numbers greater than 1, divisible only by themselves and 1).
- In the late 18th century, Carl Friedrich Gauss made an educated guess that primes appear with decreasing frequency among larger and larger numbers—but he couldn't prove it mathematically.
- Exploring the zeta function, Bernhard Riemann graphed a three-dimensional landscape in which zeros occur in a regular pattern precisely on a line that extends to infinity. He proposed that this distribution describes the pattern of prime numbers.

AVENUES FOR FURTHER LEARNING

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SPEAKING OF MATH

A Glossary for *The Story Of Math*

algebra: The branch of math that uses symbols (usually letters called variables) to represent numbers in performing calculations and expressing numbers’ relationships. *Elementary algebra*—the subject that you probably took in high school—involves solving equations with one or more variables. *Abstract algebra* involves the study of number systems or structures and the operations within them.

axiom: A statement accepted as true without proof in order to study the outcomes that arise from its application.

base number: The number of digits in any positional numbering system. In everyday life, we use a *base 10* system with the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, and 0. To express the number twelve, you write “12,” meaning 1 ten and 2 ones. In a *base 2* system, there are only two digits: 1 and 0. To express twelve, you would write “1100,” meaning 1 eight, 1 four, 0 twos, and 0 ones.

calculus: The branch of math dealing with variables that change constantly, such as velocity and acceleration. *Differential calculus* examines the rate of change (usually expressed as a slope on a graph) in particular functions as they approach a limit. *Integral calculus* deals with the limits of sums, the elements of which increase infinitely—for example, calculating the area of a circle by treating it as a polygon with an infinite number of sides.

chaos theory: The branch of math that searches for patterns or regularities in complex, dynamic, seemingly unpredictable systems in which small changes in a single variable can produce big changes in the overall system as it evolves—for example, the formation of clouds, the dispersion of water or air pollution, and the formation of crystals.

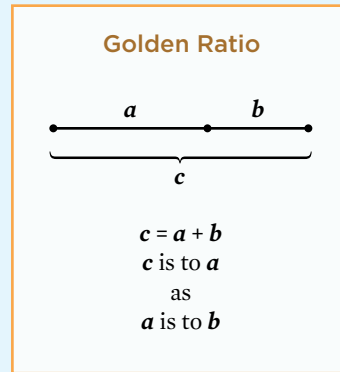
cubic equation: An equation with a variable raised to the third power, but with no exponent higher than 3 (e.g., $a + bx + cx^2 + dx^3 = 0$).

Fibonacci sequence: A specific series of numbers, in which the next number is formed by adding the two preceding numbers: 1, 1, 2, 3, 5, 8, 13, 21, etc. ($1 + 1 = 2$; $1 + 2 = 3$; $2 + 3 = 5 \dots$)

geometric sequence: An ordered string of numbers in which the ratio between any two consecutive numbers is constant. An *infinite geometric sequence* extends to infinity.

geometric sum: The additive total of the numbers in a geometric sequence.

golden ratio: An expression of proportion between two unequal quantities (such as line lengths) in which the ratio of the larger quantity to the smaller quantity matches the ratio of the sum of the quantities to the larger quantity. In other words, “the whole is to the larger as the larger is to the smaller.” Equal to approximately 1.618 and designated by ϕ , it’s used in painting, sculpture, and architecture and appears naturally in the structure of some organisms.



hypothesis: An as-yet-unproven proposition or explanation.

imaginary number: A number that doesn’t exist on the number line, but was invented by mathematicians to solve equations that involve the square root of a negative number (depicted as i). Compare *real number*.

irrational number: A number that, when expressed as a decimal, runs to infinity without repeating a pattern, e.g., π (3.141592...). Compare *rational number*.

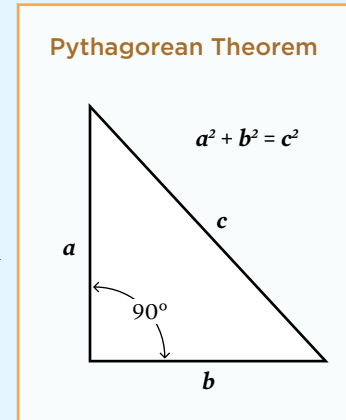
natural number: Often called *counting numbers*, natural numbers were traditionally defined as positive whole integers (1, 2, 3, 4...). However, some mathematicians began including zero in the 19th century, and to this day they have not reached a consensus about whether zero counts.

negative number: Any real number less than zero.

pi: The Greek letter (π) used to represent the ratio between a circle’s circumference and its diameter ($c = \pi d$). The formula to calculate the area of a circle is π times the radius squared (πr^2).

prime number: A positive integer greater than 1 and divisible only by itself and 1. For example, the first 10 prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23, and 29.

Pythagorean theorem: Named after the Greek mathematician Pythagoras, a theorem explaining the relationship between the lengths of the sides in any right-angle triangle. The squared length of the longest side equals the sum of the squares of the two shorter sides.



quadratic equation: An equation with a variable squared, but with no exponent higher than 2 (e.g., $ax^2 + bx = c$).

rational number: A number that, when expressed as a decimal, either stops or runs to infinity in a repeating pattern, e.g., 0.333333 . . .

real number: Any number (whether natural, rational, or irrational) that appears on the number line. Compare *imaginary number*.

theorem: A statement proven as true. Compare *hypothesis*. All theorems were once hypotheses.

topology: The branch of mathematics that studies shapes as they undergo distortion. At the most basic level, it examines the relationships among points in a geometric figure as the figure morphs. For example, all the points constituting a sphere have the same topology, even if you stretch or squash it but do not break it or cut it. For this reason, topology is sometimes referred to as “bendy geometry” or “rubber geometry.”

trigonometry: The branch of mathematics that examines the relationships among the sides and angles of triangles. It employs six functions to measure and interpret the angles in triangles: sine, cosine, tangent, cotangent, secant, and cosecant.

THE PRINCIPLES OF PROOF

How do mathematicians conclusively demonstrate a truth?

When the ancient Greeks introduced the concept of proof, they catapulted mathematics from a workaday tool into a true scientific and philosophical discipline. Math became more than just a way to calculate distance or area or weight; it became a way to discover truth in the abstract—something universal and unchanging. A mathematical proof presents statements that are general (meaning they always apply) and verifiable (meaning they can be confirmed by logic and reasoning). But how does a mathematician go about confirming a universal truth?

All mathematical proofs follow an agreed-upon form, called an axiomatic system. Whether Pythagoras was proving his theorem in the 6th century BCE, or Grigori Perelman was proving the Poincaré conjecture in the 21st century, they both followed essentially the same form. They begin with a known truth—something without proof, such as “if $a = c$, and $b = c$, then $a = b$.” Mathematicians call such statement **axioms** or **postulates**. From there, they proceed in a series of rigorous logical steps to reach a conclusion about a proposition or hypothesis. A proven hypothesis becomes a **theorem**, and it can be used (along with axioms and postulates) to prove other propositions.

Obviously, mathematicians can take various routes to get from axioms to theorems; these routes represent different types of mathematical proofs. **Direct proofs** demonstrate the truth of a hypothesis by combining axioms and theorems, sometimes without further mathematical manipulations. For example, you can show that a line bisecting the vertex formed by the two equal sides of an isosceles triangle will always produce a pair of congruent triangles. Direct proof demonstrates that the other two vertices have equal angles.

Indirect proofs, on the other hand, begin with the assumption that a proposition's opposite is true and then proceed toward certain conclusions. If one of the conclusions proves false by contradicting known axioms or theorems, then its opposite—the proposition you were trying to prove in the first place—must be true. Sometimes, logicians and mathematicians call this method **proof by contradiction** or by the Latin phrase *reductio ad absurdum* (“reduced to the absurd”). However, a certain school of mathematical thinking (known as Constructivism) doesn't accept these proofs, because they don't demonstrate the positive existence of something—only its negation.

Proofs by transposition involve “if . . . then” statements. First, you must establish what logicians call a *modus ponens* (“mode that affirms”), commonly expressed as “If p , then q .” For example, “If there's smoke, then there's fire.” Transposition demonstrates, “If not q , then not p .” In other words, “If there's not fire, then there's no smoke.”

In the 20th century, computers revolutionized one particular type of mathematical proof, known as **proof by exhaustion**. It involves dividing a hypothesis into a definite number of cases and then demonstrating each case individually. In one classic example, the 19th-century mathematician Francis Guthrie proposed what came to be known as the four-color theorem. It stated that cartographers need no more than four colors to tint the regions in any flat map so that no two adjacent regions have the same color. In 1976, a computer “proved” the theorem using over 1,500 maps. However, because computers operate independent of human understanding and rely on software that may itself prove incorrect, some mathematicians don't accept computer proofs as real proofs in the classic mathematical sense.

THE DIVINE SECRETS OF THE PYTHAGOREAN BROTHERHOOD

The story behind the ancient cult of number worshippers

They dressed only in white, took vows of silence, and abstained from meat, beans, and sex, except perhaps for reproduction. They didn't just talk to animals; they held philosophical conversations with them. They lived in communes and admitted women to their ranks as equals. They kept secrets supposedly worth killing for. Most of all, they lived by the motto “All is number.” They believed in the power of mathematics not merely to explain the world, but to transcend it.

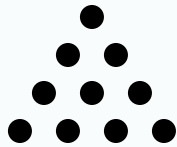
They were followers of Pythagoras of Samos, the Greek philosopher, astronomer, and mathematician. He founded a school in the southern Italian town of Croton around 525 BCE, roughly the same time that the Buddha was achieving enlightenment under a bodhi tree in India and Laozi was composing the *Daodejing* in China. Unfortunately, Pythagoras left no writings; everything we know about him comes secondhand from his acolytes, who built a religion around his teachings.

Basically, the Pythagoreans believed in numbers as the source of all reality. They thought of each individual's very being—the immortal soul, or *psyche*—as a number, which indicated the kinship of all life. Souls undergo a process called *metempsychosis*, by which they transmigrate to other physical bodies, including those of animals, by returning to earth through bean blossoms. (Naturally, this ruled out eating meat or beans.) Metempsychosis occurs again and again in an endless cycle unless the soul is purified through ascetic practices and rituals, after which it can achieve union with the divine.

If all this sounds vaguely Hindu or Buddhist, perhaps it should. Some scholars speculate that Pythagoras picked up Indian ideas from Egypt

or from the Persians, whose empire reached the Mediterranean during his lifetime. Other stories suggest more direct and personal sources. For instance, Pythagoras himself claimed to remember many of his past incarnations. In one account, he talked a bull out of eating green beans. In another, he persuaded a wild bear to stop attacking people.

Tetraktys

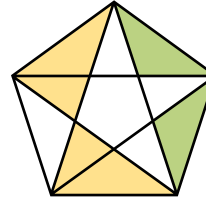


For Pythagoreans, the *tetraktys*, or “perfect triangle,” not only comprised 10 dots (a sacred number), but also revealed harmonic proportions (1:2, octave; 3:2, fifth; and 4:3, fourth).

The Pythagoreans connected math, music, and astronomy in a way that revealed a cosmic harmony—the so-called “music of the spheres.” Besides assigning genders to numerals, they attributed mystical powers to certain numbers and geometric shapes. For instance, 5 represented marriage—the union of the first female number (2) and first male (3). But they reserved their greatest awe for 10, the sum of the first four integers ($1 + 2 + 3 + 4$). The number 10 also figured in an important mystical symbol called *tetraktys* (roughly “fourness”), or the perfect triangle.

Above all, the Pythagoreans practiced strict secrecy. Converts took a vow of silence for a probationary period and never saw their master face to face. He talked from behind a full-length linen curtain while they sat without speaking or asking questions. Eventually, full members earned a personal audience, but they still called themselves *homakooi* (“those who come together to listen”). Violating the brotherhood’s code of silence brought harsh penalties. In various legends, the mathematician Hippiasus either revealed one of the brotherhood’s mathematical secrets or discovered irrational numbers, thereby threatening the basis of their theology; he was drowned for his transgression.

Pentagram



The pentagram also held special significance for Pythagoreans. Besides revealing several congruent triangles (such as those shaded yellow and green), it divided the diagonal lines of the five-pointed star into segments repeatedly showing the golden ratio.

Pythagorean sects died out, but their ideas didn’t. Great thinkers such as Copernicus and Leibniz acknowledged their debt to Pythagoras. Above all, modern scholars recognize him and his followers as the first to bring a metaphysical dimension to mathematics by understanding numbers as abstract concepts and connecting them to the physical world.

Answer: Math Magic in Art

$x = 10$, $y = 7$, and $z = 4$. The grid is a “magic square”—an arrangement of consecutive numbers in which each column, row, and diagonal adds up to the same sum, called a magic constant. (In Dürer’s example, the four corners also add up to the constant: 34.) For more math puzzles, check out “Maths in Action” in the “Mathematical Thinking” section at The Open University’s website, www.open2.net/storyofmaths.

THE PRESENTER'S STORY

Marcus du Sautoy shares his production experiences

I didn't really know a lot about the history of my subject; I always believed that what matters most is the mathematics. If you know the theorems and the proofs, is it really important who created them, or in what circumstances? Certainly the way we are taught mathematics in school reinforces this ahistorical message. So you might think that, with such a mentality, I wouldn't be the ideal candidate to present this series.

But in some ways I think that it's worked in my favor—it has become a real journey of discovery for me. Uncovering quite how much the ancient Egyptians and Mesopotamians knew about math before the ancient Greeks has been a revelation for someone brought up on the myth that it all started with Pythagoras. I was amazed to discover quite how much the Indian mathematicians of the medieval period knew about infinite series and precalculus. And visiting the places where Descartes, Fermat, Euler, and Cantor grew up brought these characters alive for me.

The programs pick up on this intellectual journey and mirror it with a real physical journey. They open with the story of math in ancient Egypt and Babylon; Cairo and the pyramids provide an exotic location for the former. Unfortunately, health and safety restrictions at the BBC stopped us from braving war-torn Iraq. So Damascus, an outpost of the Babylonian empire, became our backdrop.

The second program took us to the East and an exploration of Chinese and Indian mathematics. One of the highlights for me was the pilgrimage to Gwalior to see a tiny little temple hanging off the side of a mountain fort, big enough to fit me and a cameraman inside. We scoured the inscriptions on the walls for the earliest known example of the number zero, one of the greatest and revolutionary inventions made in India.

The mathematics of India found its way to Europe via the spice routes through central Asia. Again safety concerns denied us a trip to Iran to recreate the adventures of Omar Khayyám. So Morocco became our central Asian backdrop, where we found some fantastic horses to ride across the Atlas Mountains in my reincarnation of the great Persian. (My director informed me later that he'd decided to leave that reckless afternoon out of the health and safety report.)

Programs 3 and 4 took us to the colder climes of Europe and then on to the U.S.: a town called Descartes; Fermat Day in Beaumont-de-Lomagne, his hometown; St. Petersburg for the mighty Euler and the elusive Perelman; Göttingen for Gauss, Riemann, and Hilbert; the Nervenlinik in Halle for the unsettled Cantor; the Paris café where Bourbaki began (now a fast-food burger joint); and the Arizona desert to look for Julia Robinson's childhood haunts.

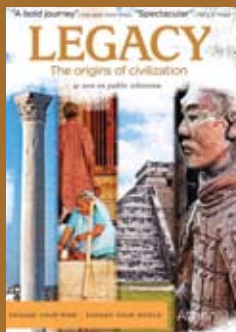
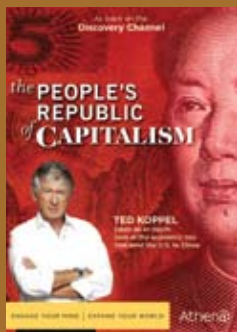
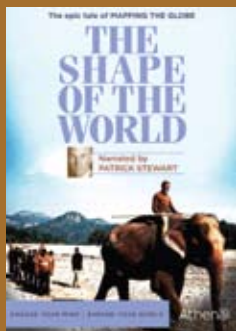
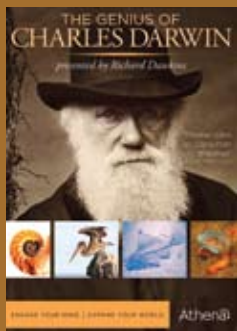
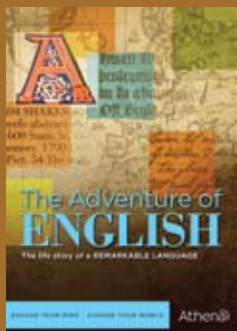
But if I had to pick one location that excited me more than any other, it has to be our one-day trip to the grey city of Kaliningrad. This is the modern name for Königsberg, the home of the seven bridges that some see as the beginning of modern topology. The city was bombed heavily during WWII, and today only three of the original bridges are left standing, while two of the others have been rebuilt. Despite the ugly nature of this modern city, I felt I was in a mathematical Disneyland. Making the journey over the bridges filled me with a childish excitement that my crew just couldn't understand. They were only too pleased to leave behind that grim skyline, but for me it was a day to treasure. It encapsulated what this whole series is about—bringing alive the stories behind the amazing intellectual journey that mathematicians have made over the last seven millennia.

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Booklet written and edited by Joseph D. Younger and Elizabeth Stocum. © 2009 Acorn Media Group Inc.

ALSO ON DVD



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